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Uncertainty Relation for Time and Energy:

We often see the uncertainty realation involving time and energy:

$\Delta E \Delta t \sim \frac{\hbar}{2}$

The question is how this should be really taken. In case of X and P, ΔX and ΔP represent uncertainties in position and momentum and the relation $\Delta X \Delta P \geq \frac{\hbar}{2}$ comes as a result of the commutation relation between X and P.

In analogy, one might think that time is also promoted to an operator in quantum mechanics with the following commotation relation with the Hamiltonian:

$[T, H] = -i\hbar$

Note the similarity between the Schrodinger equation and that for the eigenvalue problem involving the momentum operator:

$$H\,\psi(x,t) = i\hbar \frac{\partial \psi(x,t)}{\partial t}$$

$$p\,\psi(x,t) = -i\hbar \frac{\partial \psi(x,t)}{\partial x}$$

This leads to the naive guess that time is also an operator where $[T, H] = i\hbar$ (note the sign difference between the above equations).

However, according to Pauli, it turns out that this cannot be the case. If it was true, then we would have:

$$[e^{-i\alpha T}, H] = -\hbar\alpha\, e^{-i\alpha T} \qquad (\alpha \in R)$$

If $|\psi_E\rangle$ is an energy eigenstate with eigenvalue E, then we will have:

$$H\, e^{-i\alpha T}|\psi_E\rangle = e^{i\alpha T} H\,|\psi_E\rangle + \hbar\alpha\, e^{-i\alpha T}|\psi_E\rangle = (E+\hbar\alpha)\, e^{-i\alpha T}|\psi_E\rangle$$

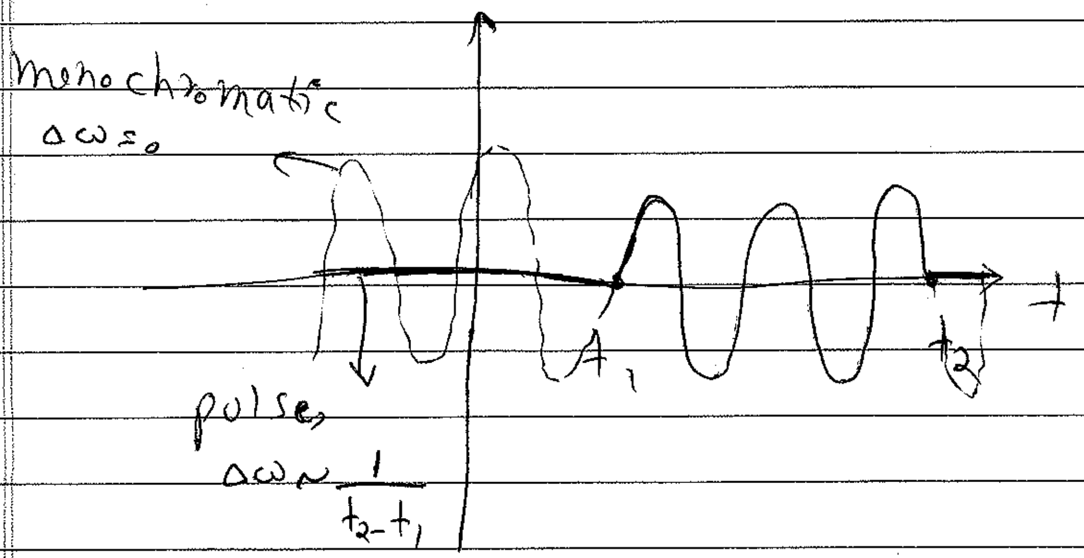
Therefore, if E is an eigenvalue of H, then $H + \hbar\alpha$ will also be an eigenvake of H. Note that α can be any real number. This implies that the spectrum of any system will be continuous. This is obviously not correct. Hence time is not an operator in quantum mechanics. Unlike position, it remains a parameter. The fact that time and position are treated differently in quantum mechanics is not surprising. After all, we are dealing with non-relativistic quantum mechanics. Without incorporating special relativity, time and space are different. It is only in relativity that "space" and "time" become "spacetime".

Now we should address the question that what the relation $\Delta E \Delta t \geq \frac{\hbar}{2}$ actually means. Here we

Consider this issue giving two interpretations.

1- An energy eigenstate has a simple dependence $\sim e^{-i\frac{Et}{\hbar}}$ on time. This resembles a monochromatic wave of frequency $\omega = \frac{E}{\hbar}$. We can learn that a wave is monochromatic by looking at its time variation (at a fixed point in space). If we observe a sinusoidal behavior, then we can say the wave is monochromatic.

However, in practice, we only observe the wave for a limited time $t_1 \leq t \leq t_2$:



Even if we see a sinusoidal wave with period $T = \frac{2\pi}{\omega}$

between t_1 and t_2, this does not mean the wave is monochromatic. It may be sinusoidal for $t<t_1$ and $t>t_2$, in which case $\Delta\omega=0$. It may also be a pulse (in the extreme limit) where there is no wave at $t<t_1$ and $t>t_2$. In the former case $\Delta\omega=0$, while in the latter case $\Delta\omega\sim\frac{1}{\Delta t}$ $(\Delta t=t_2-t_1)$. Thus, if we observe the wave for a limited time Δt, then the uncertainty in the frequency is $\Delta\omega\sim\frac{1}{\Delta t}$. We need larger Δt in order to decide about the exact value of $\Delta\omega$.

In the case of quantum mechanics, $\Delta E\sim\hbar\Delta\omega$, and hence $\Delta E\,\Delta t\sim\hbar$. The interpretation is that ΔE cannot be determined by better than $\frac{\hbar}{\Delta t}$, if we only observe the system for a limited time Δt.

2- Another interpretation follows by looking at the time evolution of a superposition of two energy eigenstates:

$$H|\Psi_1\rangle = E_1|\Psi_1\rangle \quad , \quad H|\Psi_2\rangle = E_2|\Psi_2\rangle$$

$$|\Psi_1(t)\rangle = e^{-i\frac{E_1 t}{\hbar}}|\Psi_1\rangle \quad , \quad |\Psi_2(t)\rangle = e^{-i\frac{E_2 t}{\hbar}}|\Psi_2\rangle$$

$$|\Psi(0)\rangle = a|\Psi_1\rangle + b|\Psi_2\rangle \qquad |a|^2 + |b|^2 = 1$$

$$|\Psi(t)\rangle = a e^{-i\frac{E_1 t}{\hbar}}|\Psi_1\rangle + b e^{-i\frac{E_2 t}{\hbar}}|\Psi_2\rangle \Rightarrow$$

$$|\Psi(t)\rangle = e^{-i\frac{E_1 t}{\hbar}}\left[a|\Psi_1\rangle + b e^{-i\frac{\Delta E t}{\hbar}}|\Psi_2\rangle\right]$$

Here $\Delta E = E_2 - E_1$.

For $\Delta E t \ll \hbar$, we have $e^{-i\frac{\Delta E t}{\hbar}} \simeq 1$. Therefore, $|\Psi(t)\rangle$ dependence on time is $\sim e^{-i\frac{E_1 t}{\hbar}}$. At short time, the system evolves just like an energy eigenstate with energy E_1. The second phase, $e^{-i\frac{\Delta E t}{\hbar}}$, becomes important only for $t \gtrsim \frac{\hbar}{\Delta E}$. Note that this phase is what distinguishes the superposition of two eigenstates with different energies from an eigenstate with a single energy. Therefore, we need a time $t \sim \frac{\hbar}{\Delta E}$ to figure this out. The time interval Δt required to observe the system again follows $\Delta E \Delta t \sim \hbar$.